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Integral equations depending analytically on a parameter

by

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1. Introduction

The most familiar results for integral equations consider the equation $\phi(x)+\lambda\int K(x,y)\,\phi(y)dy = f(x)$. This equation depends on the parameter λ in a particularly simple way, and a great deal is known about it, e.g. that it always has a unique solution ϕ for given f except for certain singular values λ_n of λ, and that there are only finitely many such λ_n in any compact subset of the complex plane.

Recently Lauwerier ([1], pp.429 and 430), in studying a problem that can be phrased either in terms of trigonometric series or partial differential equations, arrived at an equation of the form

$$(1)\qquad \phi(x)+\int K(x,y,\lambda)\,\phi(y)dy = f(x),$$

with the dependence on λ being analytic, but apparently not of the usual type. In this note it is shown that nearly identical results hold for (1) as for the more familiar special case in which $K(x,y,\lambda)=\lambda K(x,y)$. (We will refer to this as "the classical case".) Tamarkin [3] has given a treatment that covers a large class of equations (1), but his proof is rather complicated. That proposed below is comparatively simple, and based on a familiar idea of integral equations (approximation by "finite operators") to be found, e.g., in [2].

Consider a kernel $K(x,y,\lambda)$ for x and y in a measure space X (e.g. x and y in the real interval $a \leq x \leq b$), and λ in a connected region I of the complex plane, subject to the two conditions

(i) $\iint |K(x,y,\lambda)|^2 dx\,dy$ is continuous in I;

(ii) for each f and g in L^2,

$$\iint K(x,y,\lambda)f(x)g(y)dx\,dy$$

is analytic in I.

Here L^2 denotes the space of square-integrable functions on X. For f and g in L^2 we have the inner product

$(f,g)=\int f(x)\bar{g}(x)dx$, and the norm $\|f\| = (f,f)^{1/2}$. A set of functions u_n in L^2 is orthonormal if $(u_n,u_m)=\delta_{nm}=0$ or 1 according as $n\neq m$ or $n=m$. We assume there is a basis of such u_n, so that each f in L^2 can be written $f=\Sigma(f,u_n)u_n$.

The condition (i) above assures that for each λ in I, $K(x,y,\lambda)$ defines a completely continuous operator $K(\lambda)$ on L^2. Clearly (i) and (ii) are satisfied in the classical case where $K(x,y,\lambda)=\lambda K(x,y)$, and $\iint |K(x,y)|^2 dx\,dy < \infty$.

A number λ in I is called <u>singular</u> if and only if the equation (1) fails to have a solution ϕ in L^2 for some f in L^2. In the classical case it is well known that the set of singular values of λ is countable, and has no finite limit point. The theorem of this report states that under the more general conditions (i) and (ii) on K, the only new possibility for equation (1) is that every λ is singular.
This arrises, e.g., if u_1 and u_2 are orthonormal and $K(x,y,\lambda)= -u_1(x)\bar{u}_1(y)+\lambda u_2(x)\bar{u}_2(y)$. For in this case $\phi(x)+\int K(x,y,\lambda)\varphi(y)dy= \varphi(x)-u_1(x)\,(\varphi,u_1)+\lambda u_2(x)\,(\varphi,u_2)$ is orthogonal to u_1, so that $\varphi+K\varphi = f$ fails to have a solution for $f = u_1$.

2. The theorem

Here we state and prove the main result. At a certain stage it is necessary to derive the analyticity of the n^{th} power $R(\lambda)^n$ of an operator $R(\lambda)$ from the analyticity of $R(\lambda)$. This is justified by the lemma proved in section 3.

<u>Theorem</u>. If $K(x,y,\lambda)$ satisfies conditions (i) and (ii), then in the connected region I either every λ is singular, or the singular values of λ have no finite limit point in I.

<u>Proof</u> We will assume that λ_0 in I is a finite limit point of the set of singular values for (1), and show that every λ in I is singular.

Let $\{u_n\}$ $(n=1,2,\dots)$ be an orthonormal basis for L^2, and set

$$a_{nm}(\lambda)=\iint K(x,y,\lambda)\bar{u}_n(x)\;u_m(y)dx\,dy$$

so that in the sense of mean convergence

$$K(x,y,\lambda)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}(\lambda)\ u_n(x)\ \bar{u}_m(y).$$

Because of condition (ii), the a_{nm} are analytic in I. We define the kernel

$$K_N(x,y,\lambda)=\sum_{m=1}^{N}\sum_{n=1}^{N} a_{nm}(\lambda)u_n(x)\ \bar{u}_m(y).$$

Denote the operators with the above kernels by $K(\lambda)$ and $K_N(\lambda)$ respectively, and define

$$R_N(\lambda) = K(\lambda)-K_N(\lambda).$$

For any functions g and h in L^2, we have set $(g,h)=\int g\bar{h}$. Then, by (ii), $(K(\lambda)g,h)$ is analytic in I; and because the a_{nm} are analytic, $(K_N(\lambda)g,h)$ is analytic in I. Hence $(R_N(\lambda)g,h)$ is analytic in I. In view of the lemma proved below, we conclude further that $(R_N(\lambda)^n g,h)$ is analytic in I, for each n,g and h.

Suppose λ_o is a finite limit point in I of singular values for (1), and choose N so that $\iint|K(x,y,\lambda_o)-K_N(x,y,\lambda_o)|^2 dx\ dy < 1/2$. The continuity of the a_{nm} and condition (i) then guarantee that for some $S > 0$ we have that $\{|\lambda-\lambda_o|< S\}$ belongs to I, and

$$(2)\quad \|R_N(\lambda)\|\leq\iint|K(x,y,\lambda)-K_N(x,y,\lambda)|^2 dx\ dy < 1/2$$

for λ in $C = \{|\lambda-\lambda_o|< S\}$. Here $\|A\|$ denotes the norm of the operator A on L^2. In (1) let $\phi+R_N(\lambda)\phi=\psi$, a reversible substitution by virtue of $\phi = (\sum_{o}^{\infty}[-R_N(\lambda)]^n)\psi = B(\lambda)\psi$. This series for $B(\lambda)$ converges in norm because of (2). From the analyticity of $(R_N(\lambda)^n g,h)$ and the uniform convergence of the series $(B(\lambda)g,h)=\sum_{o}^{\infty}([-R_N(\lambda)]^n g,h)$, it follows that $(B(\lambda)g,h)$ is analytic.

With the above substitution, equation (2) becomes

$$(3)\quad \psi+K_N(\lambda)B(\lambda)\psi = f.$$

This can be reduced to a system of equations in N unknown constants by the following familiar device. Let $\sum_{n=1}^{N}a_{nm}(\lambda)u_n(x)=v_m(x,\lambda)$, and $B^*(\lambda)u_m(y)=w_m(y,\lambda)$, with B^* the adjoint of B, so that (3) becomes

$$(4)\quad \psi(x)+\sum_{1}^{N} v_m(x,\lambda)\int \overline{w}_m(y,\lambda)\psi(y)dy=f(x),$$

whose solution amounts to finding the numbers $c_m=\int \overline{w}_m(y,\lambda)\,\psi(y)dy$. To do this, take the inner product of (4) with each of the w_k to get the system

$$(5)\quad c_k+\sum_{m=1}^{N}(v_m,w_k)c_m = (f,w_k),\quad k=1,\dots,N.$$

It is easy to check that any solution ψ of (4) yields a solution $c_k = (\psi,w_k)$ of (5), and conversely any solution c_k of (5) yields a solution $\psi = f-\sum_{1}^{N} c_k v_k$ of (4). Thus (4) is solvable for every f if and only if (5) is. Since B^* is invertible, the $w_k= B^*u_k$ are linearly independent; thus the (f,w_k) can be taken as any N numbers by the proper choice of f. It follows that (5) has a solution for all f if and only if λ is not a zero of

$$(6)\quad \det\left(\delta_{mk}+\int v_m(x,\lambda)\overline{w}_k(x,\lambda)dx\right).$$

It follows from the form of the v_m and w_k and the analyticity of $(B(\lambda)g,h)$ that the function (6) is analytic for $|\lambda-\lambda_o|<S$. Since λ_o is a limit of singular values of λ, (6) vanishes on a set with limit point λ_o, hence (6) vanishes identically for $|\lambda-\lambda_o|<S_o$. It follows that all these values of λ are singular. It is clear that this result can now be extended to all of I, since I is connected.
Thus the proof is complete modulo the lemma referred to above.

3. <u>The lemma</u>

Before stating the lemma, we introduce the following notation in connection with the operator A with kernel $A(x,y)$: $|||A|||^2=\iint |A(x,y)|^2dx\,dy$. Thus condition (i) above is that $|||K(\lambda)|||$ is continuous.

<u>Lemma</u> Let $A(\lambda)$ and $B(\lambda)$ be integral operators defined for λ in a region D, and such that in D $|||A(\lambda)|||$ and $|||B(\lambda)|||$ are bounded, and $(A(\lambda)f,g)$ and $(B(\lambda)f,g)$ are analytic in D for each f and g in L^2. Then $A(\lambda)B(\lambda)$ has the same properties.

<u>Proof</u> It is clear that $|||A(\lambda)B(\lambda)|||$ is bounded in D, since $|||A\,B||| \leq |||A|||\cdot|||B|||$. To show the required analyticity, we expand A and B in terms of the operators P_{nm} defined in terms of the orthonormal base $\{u_n\}$ by $P_{nm}u_k=\delta_{km}u_n$, with δ_{km} the Kronecker

delta. Then if $f = \sum f_n u_n$ and $g = \sum g_m u_m$, we have $(P_{nm} f, g) = f_m \bar{g}_n$. If the kernel of $A(\lambda)$ is $A(x,y,\lambda) = \sum_{n,m} a_{nm}(\lambda) u_n(x) \bar{u}_m(y)$, then $A(\lambda) = \sum_{n,m} a_{nm}(\lambda) P_{nm}$, and $|||A(\lambda)|||^2 = \sum |a_{nm}(\lambda)|^2$. In the same way we set $B(\lambda) = \sum b_{\nu\mu}(\lambda) P_{\nu\mu}$. Since $P_{nm} P_{\nu\mu} = \delta_{m\nu} P_{n\mu}$, we have

$$(A(\lambda)B(\lambda)f,g) = \sum_{n,m,\nu} a_{n\nu}(\lambda) b_{\nu m}(\lambda)(P_{nm} f, g)$$

$$= \sum_{n,m,\nu} a_{n\nu}(\lambda) b_{\nu m}(\lambda) f_m \bar{g}_n .$$

Because of the analyticity of $(A(\lambda)u_m, u_n)$ and $(B(\lambda)u_\mu, u_\nu)$ each term of this series is analytic in D. Thus in view of Vitali's convergence theorem it suffices to show that $\sum |a_{n\nu}(\lambda) b_{\nu m}(\lambda) f_m \bar{g}_n|$ is uniformly bounded in D. But a few applications of the Schwartz inequality yield $\sum |a_{n\nu}(\lambda) b_{\nu m}(\lambda) f_m \bar{g}_n| \leq |||A(\lambda)||| \cdot |||B(\lambda)||| \cdot \|f\| \cdot \|g\|$, which with the assumed boundedness of $|||A(\lambda)|||$ and $|||B(\lambda)|||$ on D completes the proof.

It is clear how this lemma yields an inductive proof that $(R_N(\lambda)^n g.h)$ is analytic for n=1,2,.... .

References

[1] H.A. Lauwerier, "On Certain Trigonometrical Expansions", Journal of Mathematics and Mechanics, vol.8 (1959) pp. 419-432.

[2] S.G. Mikhlin, Linear Integral Equations, Delhi, 1960. (Translated from a Russian edition of 1959).

[3] J.D. Tamarkin, "On Fredholm's Integral Equations Whose Kernels are Analytic in a Parameter", Annals of Mathematics, 2nd series, vol. 28 (1926-27), pp. 127-152.